# Convective, molecular and total momentum flux tensors

Let us discuss what is momentum flux?

Momentum flux = Momentum flow per unit area per unit time.

$$= \frac{\text{Momentum}}{m^2 \ sec.}$$

What is momentum by the way?

You learn this in your "+2" physics for solid particles in motion.

For a particle of mass "$m$" and velocity "$v$", the momentum is = "$mV$".

"Fluid dynamics is about momentum transport"

For the case of fluid, we show how to develop the expressions for the convective momentum flux, first.

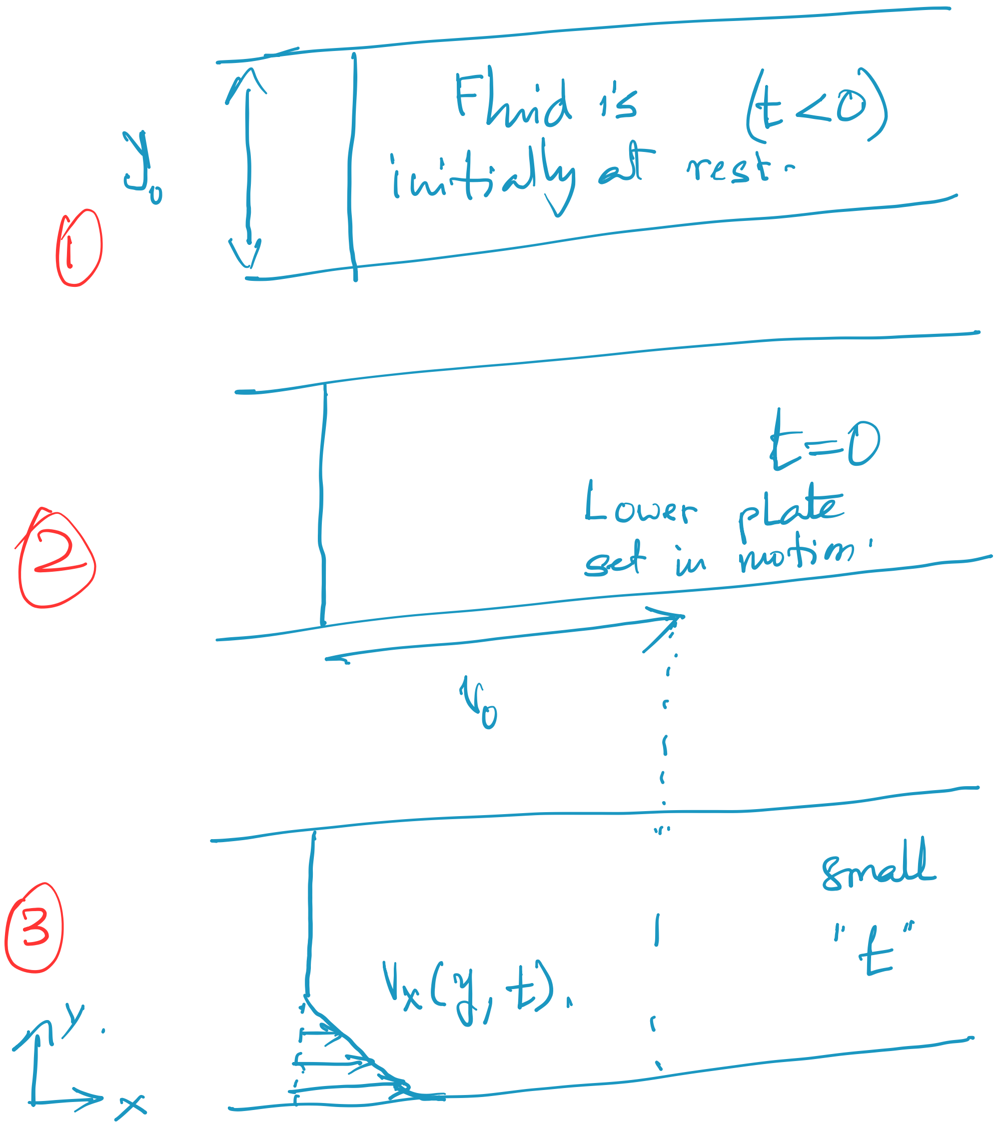
We consider two cases.

(a) Unidirectional shear flows

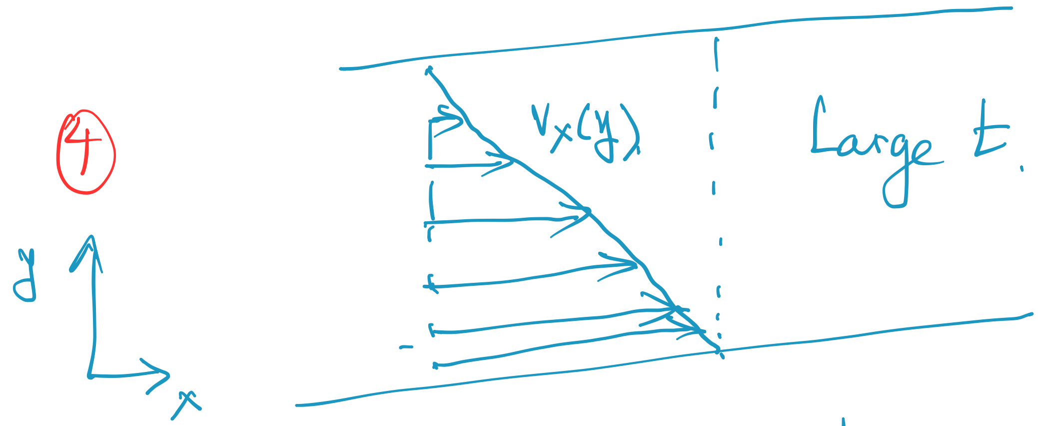
(b) General three-dimensional flows

## (a) Unidirectional shear flows:-

Consider a simple flow system.



4




Final velocity distribution in steady flow.

The volumetric rate of flow of fluid in the x-direction through an area element "$\Delta y \Delta z$" is

$$= v_x \Delta y \Delta z. \quad = m^3/sec.$$

Rate of convective flow of x-momentum through an element of area $\Delta y \Delta z$ is,

$$= \frac{\text{volume of fluid}}{\text{time}} * \frac{x\text{- momentum}}{\text{volume of fluid}}$$

$$= \frac{x\text{-momentum}}{\text{time}} = (v_x \Delta y \Delta z)\, \rho\, v_x.$$

Divide this by $\Delta y \Delta z$ to get the flux

Convective flux of $x$-momentum in the $x$-direction.

$$\pi^{(c)}_{xx} = \rho\, v_x v_x.$$

Dimensions are of $\frac{\text{momentum}}{\text{area}\cdot\text{time}}$.

throughout our course, flux of any "quantity" has the dimensions of $\frac{\text{"quantity"}}{\text{area}\cdot\text{time}}$.

Digression:- (Refer to chapter-0)

In the case of discussion on molecules/atoms, we discuss masses of molecules etc., In our subject, instead of that we talk about mass of material inside a "tiny" region "$\Delta x \Delta y \Delta z$" containing N molecules.

$$\text{Density} \Rightarrow \rho = \frac{\sum_{i}^{N} m_i}{\Delta x \Delta y \Delta z} \; [=] \; M L^{-3}$$

$$\text{velocity} \Rightarrow \underline{v} = \frac{\sum_{i}^{N} v_i}{N} \; [=] \; L T^{-1}$$

(velocity is a vector). $\underline{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$.

$$\text{Speed} = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

Rate of volume flow through $\Delta y \Delta z$

$$= v_x \Delta y \Delta z \; [=] \; L^3/T$$

Rate of mass flow through $\Delta y \Delta z$.

$$= \frac{\text{Mass}}{\text{volume}} \times \frac{\text{volume}}{\text{time}}$$

$$= \rho \left( v_x \Delta y \Delta z \right) [=] \frac{M}{T}$$

Rate of momentum flow

$$= \frac{x\text{-momentum}}{\text{volume}} \times \frac{\text{volume}}{\text{time}}.$$

$$= \rho v_x (v_x \Delta y \Delta z) \; [=] \; \frac{ML}{T^2}$$

Rate of kinetic energy flow

$$= \frac{1}{2} \rho v_x^2 \cdot v_x \Delta y \Delta z \; [=] \; \frac{ML^2}{T^3}$$

Divide these quantities by $(\Delta y \Delta z)$
area.

Flux of mass. $= \rho v_x.$

Flux of momentum $= \rho v_x v_x$

Flux of kinetic energy $= \frac{1}{2} \rho v_x^2 v_x.$

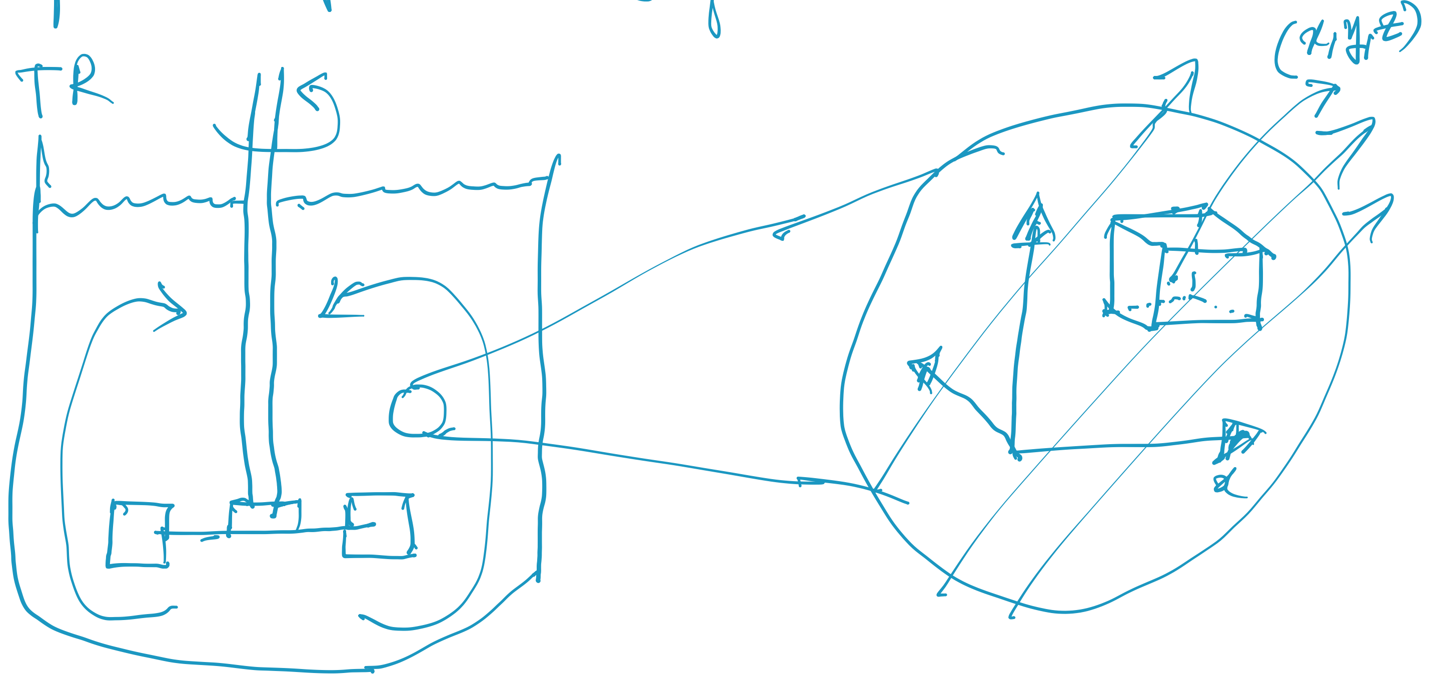
Flux of Internal energy $= \rho \hat{U} v_x = MT^{-3}.$

Coming back to convective transport.

## (b) General 3-D time dependent flow :-

We are generally interested in a much more complicated system than simple 2-plate configuration such as,

CSTR




Velocity components depend on $x, y, z$ & $t$

$v_x(x, y, z, t)$

$v_y(x, y, z, t)$

Across the plane perpendicular to the x-direction.

$(\Delta y \Delta z)$.

We have. $v_x(\rho \underline{v})$.

$$\Pi_{xx}^{(c)} = \rho v_x v_x, \quad \Pi_{xy}^{(c)} = \rho v_x v_y, \quad \Pi_{xz}^{(c)} = \rho v_x v_z$$

Across the plane perpendicular to the y-direction,

$(\Delta x \Delta z)$. $\quad v_y(\rho \underline{v})$.

$$\Pi_{yx}^{(c)} = \rho v_y v_x, \quad \Pi_{yy}^{(c)} = \rho v_y v_y, \quad \Pi_{yz}^{(c)} = \rho v_y v_z$$

Across the plane perpendicular to the z-direction.

$(\Delta x \Delta y)$. $\quad v_z(\rho \underline{v})$

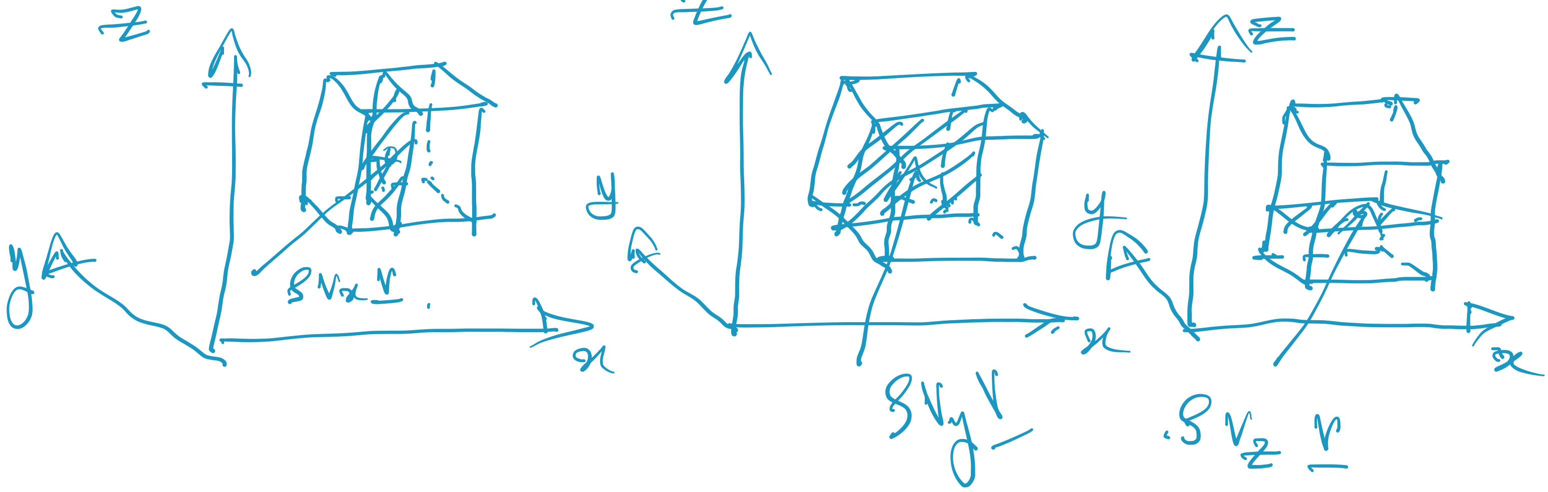
$$\Pi_{zx}^{(c)} = \rho v_z v_x, \quad \Pi_{zy}^{(c)} = \rho v_z v_y, \quad \Pi_{zz}^{(c)} = \rho v_z v_z.$$

Denote this in general by

$$\Pi_{ij}^{(c)} \quad \& \quad \rho v_i v_j$$

the first subscript gives the direction of transport and the second gives the component of momentum/velocity that is being transported.

These quantities with two subscripts are known as tensors and have total 9 components.




## Summary

| Direction normal | Flux of momentum | Components wise x | y | z |
|---|---|---|---|---|
| x | $\rho v_x \underline{v}$ | $\rho v_x v_x$ | $\rho v_x v_y$ | $\rho v_x v_z$ |
| y | $\rho v_y \underline{v}$ | $\rho v_y v_x$ | $\rho v_y v_y$ | $\rho v_y v_z$ |
| z | $\rho v_z \underline{v}$ | $\rho v_z v_x$ | $\rho v_z v_y$ | $\rho v_z v_z$ |

$$\Pi^{(c)} = \rho \underline{v}\,\underline{v}.$$

## Molecular momentum tensor

### Newton's law

Convective momentum flux refers to how momentum is transported by being swept along by the bulk motion.

We now focus on molecular momentum flux:-

in (a) unidirectional shear flow

(b) General three dimensional flow.

(a)

Now we will consider the same flow configuration as before.

At $t=0$, the lower plate is set into motion in +ve $x$-direction.

As time increases, the fluid near the lower plate gains momentum, and then successive layers of fluid in the $y$-direction gain momentum.

Ultimately, a linear steady state profile is achieved as shown in the figure.

When the steady motion has been attained, a constant force is required to maintain the motion of the lower plate.

$$\frac{F}{A} = \mu \frac{v_0}{y_0}.$$

The above form is found by experimentalists. (Newton?)

what they saw is that the force.

is $\propto$ Area

$\propto v_0$

$\propto 1/y_0$. for many fluids. (including water and air).

They found that the proportionality constant "viscosity" is different for different fluids.

Replace $\frac{F}{A}$ by $\tau_{yx}$

$\tau_{yx}$ = Force in the x-direction acting on unit area perpendicular to y-direction.

Replace $\frac{v_0}{y_0}$ by "$-\frac{dv_x}{dy}$" because $\frac{dv_x}{dy} < 0$

So,
$$\boxed{\tau_{yx} = -\mu \frac{dv_x}{dy}} \rightarrow (1)$$

"Shear stress is proportional to the negative of velocity gradient" is known as Newton's Law.

This is found to be very useful for many simple fluids. These are Newtonian fluids.

"Polymeric fluids, fluids with particles, other complex fluids" don't follow this law.

$\tau_{yx}$ = "Flux of $x$ momentum in the positive $y$-direction"

Two interpretations gave us interchangeable names for $\tau_{yx}$

Viscous shear stress.

Viscous momentum flux.

The second name suggests that momentum naturally flows from a region of high velocity to a region of low velocity. (much like temperature).

Just like temperature gradient is driving force for heat transfer, the velocity gradient is the driving force for "molecular momentum transport".

## (b) General 3D flow

We need to generalize ① for a very general flow, where velocity field has three components.

$$v_x(x, y, z, t)$$
$$v_y(x, y, z, t)$$
$$v_z(x, y, z, t)$$

This generalization took "150 years". So beginners don't need to think that it is obvious.

It is much better to write down