Dealiasing of quadratic non-linearity

Aliasing

To understand the aliasing phenomena due to quadratic non-linearities, we will consider the following simple example where there exist only four wave numbers.

Consider the two functions:

$$u = \sin(x) + 2\sin(2x) + 3\sin(3x) + 4\sin(4x) \tag{1}$$

$$v = \cos(x) + 2\cos(2x) + 3\cos(3x) + 4\cos(4x) \tag{2}$$

The Fourier amplitudes corresponding to these four modes for both the functions are given by $(1, 2, 3, 4)$. Only that the first one contains all Sines and the second one contains all Cosines.

It is easy to see that the product of these two contain higher harmonics i.e., $(5, 6, 7, 8)$:

$$\begin{aligned} uv = \frac{1}{2}(&\sin(2x) + 4\sin(3x) + 10\sin(4x) + 20\sin(5x) + \\ &25\sin(6x) + 24\sin(7x) + 16\sin(8x)) \end{aligned} \tag{3}$$

We will have a problem if we consider discretized version of our function. Let us say on $x \in (0, 2\pi)$, We have

The above simple example made it clear that there will be incorrect amplitudes for low wave number modes if we are not going to rectify the issue of aliasing arising due to quadratic non-linearities.

Aliasing instability: Generalization

We consider the inviscid Burgers equation as an example of nonlinear PDE with the quadratic non-linearity.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{4}$$

We may consider the arbitrary initial condition, say $u(x, t = 0) = \sin x$ for $x \in (0, 2\pi)$, and want to find out the solution for all other times.

Note that the second term on the right side is a quadratic non-linearity in u. We want to use the pseudo-spectral method to solve this problem. We need to use de-aliased version of the pseudo-spectral method as this involves the quadratic non-linearity.

In what follows, we will understand why we need the dealiasing of this quadratic non-linearity.

Note that, as part of the pseudo-spectral method, we represent the solution as a sum of large number of Fourier modes and let them evolve with time. We make use of discrete Fourier Transform (DFT) and the inverse discrete Fourier Transform.

Discrete set of points (N of them with spacing of h) in the real space : $x \in \{h, 2h, ..., 2\pi - h, 2\pi\}$.

In Fourier space : $k \in \{-N/2+1, -N/2+2, ..., N/2\}$

The formula for the DFT is:

$$\hat{u}(k_j) = h \sum_{n=1}^{N} u(x_n) e^{-ik_j x_n} \qquad k_j = -N/2+1, ..., N/2, \tag{5}$$

and the inverse DFT is given by

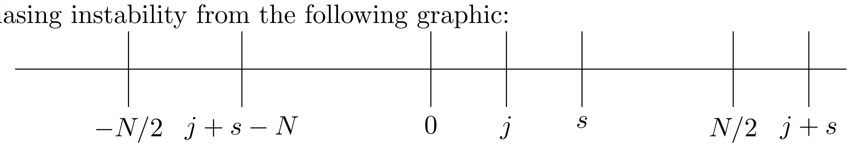
$$u(x_n) = \frac{1}{2\pi} \sum_{k_j=-N/2+1}^{N/2} \hat{u}(k_j) e^{ik_j x_n} \qquad n = 1, ..., N, \tag{6}$$

Consider a term of the form $u(x)v(x)$ in a non-linear PDE. For the case of Burgers equation, $v(x)$ is $\frac{\partial u}{\partial x}$.

For such a term, we have:

$$[u(x_n)v(x_n)] = \sum_{k_j=-N/2+1}^{N/2} \hat{u}(k_j) e^{ik_j x_n} \sum_{k_s=-N/2+1}^{N/2} \hat{v}(k_s) e^{ik_s x_n} \tag{7}$$

Note that there are a total of N Fourier modes. This finite sum now leads to the aliasing instability in the method. We can understand the origin of this aliasing instability from the following graphic:



The truncation of the Fourier modes beyond the $N/2$ implies that we now only consider the evolution of Fourier modes between $-N/2$ and $N/2$. However due to interaction of two modes, via the quadratic nonlinearity (see equation 7), implies that the resulting mode possibly can turn out to be outside $(-N/2, N/2)$ as shown in the above graphic (shown as $j+s$ mode due to interaction of j and s modes). However, this gets aliased to $j+s-N$ mode due to periodic nature of exponential function (shown in the graphic), thus leading to aliasing instability. Thus, we need to make that the modes lying beyond $(-N/2, N/2)$ do not get aliased to a 'lower' mode in the interval $(-N/2, N/2)$. This requires us to use the '2/3 rule' which is the method to de-alias the quadratic nonlinearities, and is derived next. In this method, we filter the modes beyond a certain wavenumber to make sure the 'high' wavenumber modes are not aliased to 'low' wave number modes.

$\frac{2}{3}$ rule for dealiasing:

This rule sets the amplitudes of the modes beyond a wavenumber to zero. Let us say this wavenumber is K. Let modes j and s be in the interval $(0, K)$.

If $j+s > N/2$ (outside range), then the amplitude corresponding to $j+s$ will be aliased to $j+s-N$ as mentioned earlier. We demand that $j+s-N < -K$ in the not used part of the spectrum. The largest j and s in the range are $j = s = K$.

Thus,

$$\begin{aligned} j+s-N &= 2K-N \\ 2K-N &< -K \\ 3K &< N \\ K &< N/3 \end{aligned}$$

This sets the threshold $K = N/3 = 2/3(N/2) = (2/3) * k_{max}$ leading to the Famous Orszag's $\frac{2}{3}$rd rule for the pseudo-spectral method.

This rule can be used for solving any nonlinear partial differential equation that only involves quadratic non-linearities such as Navier-Stokes equations. Rules may be derived in similar manner for any other algebraic non-linearities such as cubic and quartic non-linearities.